A Theory for the Most Stable Variable Viscosity Profile in Graded Mobility Displacement Processes

The stability of the displacement of a viscous fluid in a porous medium by a less viscous fluid containing an additive is considered. For the case in which the total amount of injected additive is fixed, a theory has been formulated to find the graded mobility profile which minimizes the instability due to an adverse mobility ratio. The result is a constrained nonlinear eigenvalue problem in which the mobility ratio of the fluids, α, and the total dimensionless amount of additive, N, appear as parameters. Estimates show N is typically large for practical situations. An asymptotic solution for large N is developed which shows that the optimal mobility profile is always an exponential curve at leading order. It is shown that although the optimal mobility profile is nearly exponential, the optimal concentration profile is not necesarily so, contrary to previous suggestions in the literature. Furthermore, it is shown that, unlike the case of a mobility jump, a graded mobility process under optimal conditions has amplifications independent of displacement velocity. We also discuss a possibile stabilization mechanism due to the very long wavelength disturbances which occur under optimal conditions.

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SCOPE

The objective of this work is to provide a theoretical analysis of the stability properties of displacement processes in porous media when an additive such as a polymer is used to create a smooth variation in the mobility of the displacing phase. Most if not all enhanced oil recovery processes involve instabilities which are driven by mobility differences between the phases, and many of them seek to reduce the severity of the instability by the addition of polymers, foams, and other additives which decrease the mobility of the displacing phase. Thus, our work is generally relevant to the mathematical description of the stability of such processes. Similar considerations may apply to fixed-bed regeneration processes and to filtration processes. Previous work on the subject has to a large extent been empirical and experimental, with little or no theoretical framework within which to interpret results. Our work is intended to provide such a framework.

CONCLUSIONS AND SIGNIFICANCE

We have shown that the degree of stabilizaton achieved by addition of mobility-decreasing substances depends primarily on a dimensionless parameter, N, which is proportional to the quantity of the substance. We show that to achieve any appreciable stabilization, N must typically be large. Given this, we then prove under fairly general conditions that the mobility profile which minimizes the growth rate of the instability is always exponential in space. Furthermore, the growth rate is given asymptotically for large N as $\sigma \sim f(\alpha)/N$ where $f(\alpha)$ is a function of the mobility ratio, α, of the two pure phases. The function of α depends upon the details of the relationship between mobility and additive concentration. Such exponential mobility profiles may or may not imply exponential concentration profiles, a fact often overlooked in the literature. Our results are among the first which attempt to put the design of such processes on a firm mathematical footing.

INTRODUCTION

Oil recovery processes in which a fluid such as water is used to displace oil are used to varying degrees. The efficiency of these processes is limited, because a less viscous displacing fluid often fingers through the oil, leading to bypassing and a decrease in total oil recovery. Similar effects may occur in fixed-bed regeneration and other analogous processes. In a porous medium the fingering process is related to the stability of a convected macroscopic interface between oil and a displacing fluid. This problem was initially studied by Chuoke et al. (1959), who assumed that dis-

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placement of one phase by the second was complete; i.e., they ignored relative permeability and partial saturation effects. They showed the displacement under these conditions to be stable for mobility ratios less than one and unstable otherwise. Recently, Jerauld et al. (1984) and Yortsis and Huang (1984) have considered the effects of mobility profiles (occurring as a result of partial saturation and relative permeability effects) on the stability of linear displacement. These authors find some degree of stabilization, in the sense that the critical mobility ratio is raised above unity. The critical value of this parameter depends upon the form of the relative permeability curve.

In this paper we concentrate on the mobility changes due to additives, and not on those due to so-called fractional flow effects. These latter effects can become important for mobility ratios near unity, however, and can easily be incorporated into the theory if necessary. For a review of work on this class of problems through 1975, see Wooding and Morel-Seytoux (1976).

Since the instability is driven by viscosity differences, attempts are often made to stabilize the flow by increasing the local viscosity of the displacing fluid near the oil interface. This can be accomplished if additives are used. The idea of using varying viscosity (concentration) profiles in oil displacement processes is not new, and graded mobility processes have been used in both laboratory and field experiments. Some of the previous literature is reviewed in our earlier paper (Gorell and Homsy, 1983).

Past work of relevance here includes that of Mungan (1971), who performed experiments in porous media and two-dimensional Hele-Shaw cells using polyacrylamide and polyethylene oxide additives. By properly grading the concentration profile, complete stabilization of displacements with unfavorable viscosity ratios were accomplished. The concentration profile used was one which varied exponentially with distance from the front. The length scale for this graded bank was determined in an ad hoc fashion.

Uzoigwe et al. (1974) undertook two-dimensional numerical simulations of the graded concentration process in which the stability of various injected profiles was examined and the least unstable profile was determined by trial and error. Over 100 separate computer runs were performed. The conclusions of interest here were that the concentration must be decreased smoothly from a value resulting in a near match in viscosity at the front, and the concentration should fall to zero in a finite distance. The best profile seemed to be one in which the concentration decreased exponentially with distance.

The indication from these studies is that the stability of porous media displacements depends both on the amount of additives used and the manner in which they are injected. In our previous work (Gorell and Homsy, 1983), we formulated an optimization problem to find the viscosity, or equivalently the concentration profile, which minimizes the effects of the instability. This was referred to as the optimal profile. The result is a nonlinear constrained eigenvalue problem which must be solved for the optimal profile and the growth constant of the instability. The parameters are the ratio of the displaced to displacing fluid viscosities, α, and the total amount of injected additive, N, where

$$N = \left(\frac{(\mu_2 - \mu_1)U}{T}\right)^{1/2} \frac{M}{A} \frac{\left(\frac{d\mu}{dc}\right)_{c=0}}{\mu_1} \tag{1}$$

Here μ is the viscosity divided by the permeability of the medium, or the inverse of the mobility; U is the displacement velocity; T is the effective interfacial tension; M is the total moles of additives; A is the cross area of the flow; c is the concentration; and $d\mu/dc$ is the variation of viscosity with concentration. The subscripts *1,2* refer to the displacing and displaced fluid, respectively. N is a dimensionless quantity which arises when the length scale characteritic of the problem without additives is used in the nondimensionalization of the problem. In addition to the parameters α and N, the functionality between the additive concentration and the viscosity of the resulting mixture must be specified. In our previous work, the problem was solved over the entire range of N for the simple case when the viscosity is a linear function of concentration.

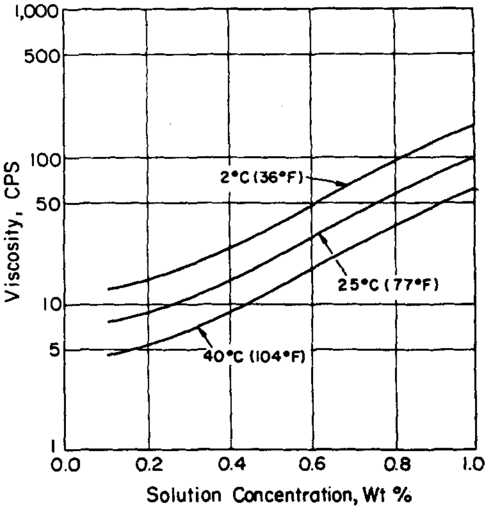

Figure 1. Viscosity data for polyacrylamide (Dow Separan NP10) in water.

Although a linear concentration-viscosity relationship facilitates the solution of the problem, it may not be realistic for actual additives. As an example, in Figure 1 we present a concentration-viscosity curve, obtained from the Dow Company, for a form of polyacrylamide (Dow Separan NP10 Polymer) in water. Since a semilogarithmic plot shows near linearity, it is in this case more appropriate to represent the viscosity as a nearly exponential function of concentration. Data on other polymer solutions show a similar strong concentration dependence. Other additives, e.g., foams, would also show such complex behavior. One of the objectives of the present work is to develop a theory which holds for an arbitrary relationship between viscosity and concentration.

To achieve a high degree of stabilization, large amounts of additives are typically used. Although field data with which to calculate an accurate value of N are not readily available, it is possible to estimate its magnitude to find that it is typically large. For example, consider a water-polyacrylamide system used to displace oil with a viscosity of 0.05 Ns/m^2. If nominal values of the other parameters are taken as $T \sim 10^{-2} N/m$, $U \sim 3 \times 10^{-4}$ cm/s, $\mu_1/(d\mu/dc) \sim 2 \times 10^{-3} g/cm^3$, porosity of 0.2, and if 0.5 pore volumes over a 30 m length are filled with polymer of average concentration of 10^{-3} g/cm^3, then $N \approx 2{,}500$.

When N is large, it is possible to perform an asymptotic analysis and obtain results for the optimal profile. The general problem as formulated in our previous paper remains unchanged. In this paper, we consider N to be large and extend our previous results to general relations between concentration and viscosity. Details of the large-N solution giving the minimum growth constant, most dangerous wave number, and the injection length scale will also be presented.

PROBLEM FORMULATION

The prototype problem for the linear stability of a polymer-type flood is pictured in Figure 2, where oil with viscosity μ_2 is displaced with a fluid (usually water) having viscosity μ_1, and additives are used to increase the viscosity in a prescribed manner $\mu(x)$ over a

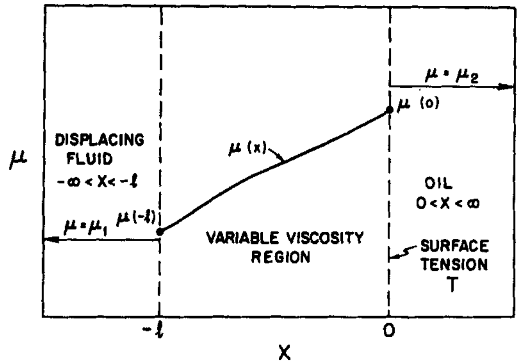

Figure 2. Schematic of viscosity profile sequence in a displacement process.

length l near the interface. As shown, three regions arise in the analysis: pure oil, followed by a variable viscosity mixture of length l, and finally the pure displacing fluid. It is assumed that the viscosity variation in the mixture is attained by varying the solute concentration.

The detailed development of this problem has been given in Gorell and Homsy (1983); here we only highlight the formulation. We consider a continuum model for porous media flow throughout this paper. The base state for the stability analysis is a constant, steady, one-dimensional flow with velocity U in the x_1 direction. Transferring to the convected coordinate $x = x_1 - Ut$, the perturbed displacement velocity in the x direction is u', which can be represented as

$$u' = \Psi(x) \exp\{iky + \sigma t\}. \tag{2}$$

It is assumed that the fluids are neutrally buoyant and incompressible, the medium is homogeneous, and dispersion, diffusion, and adsorption can be neglected. Combining Eq. 2 with the continuity equation, Darcy's law, and the equation for species continuity results in a second-order ordinary differential equation for $\Psi(x)$; cf. Eq. 5 below.

The length and time scales are nondimensionalized with respect to $1/\sqrt{3}k_m$ and $2/(3\sqrt{3})\sigma_m$ respectively, where

$$\frac{1}{\sqrt{3}k_m} = \left(\frac{T}{(\mu_2 - \mu_1)U}\right)^{1/2} \tag{3}$$

$$\frac{2}{3\sqrt{3}\sigma_m} = \left(\frac{\mu_1 + \mu_2}{(\mu_2 - \mu_1)_U}\right)\left(\frac{T}{(\mu_2 - \mu_1)U}\right)^{-1/2} \tag{4}$$

Here T is the effective interfacial tension which acts at the oil interface ($x = 0$). σ_m and k_m are the maximum growth constant and wave number corresponding to the constant viscosity problem. Viscosities are nondimensionalized with respect to μ_1.

Boundary conditions for $\psi(x)$ are obtained by requiring u' to be continuous and balancing the pressure drop with an effective surface tension at an interface. If $\alpha \equiv \mu_2/\mu_1$ then the dimensionless differential equation and boundary conditions for $\psi(x)$ are

$$\mu(\psi'' - k^2\psi) + \mu'\psi' + \mu'\beta\psi = 0 \tag{5}$$

$$\mu(0)\psi'(0) = \{-k\alpha + \beta[\alpha - \mu(0)] - (\alpha - 1)k^2\beta\}\psi(0) \tag{6}$$

$$\psi'(-L) = k\psi(-L) \tag{7}$$

$$\mu(-L) = 1 \tag{8}$$

where

$$\beta \equiv \left(\frac{\alpha + 1}{\alpha - 1}\right)\frac{k^2}{\sigma}, \tag{9}$$

and the prime denotes differentiation. The parameter β may be considered an inverse growth constant of the instability.

For a given amount of additive, the viscosity profile which minimizes the maximum growth constant is desired. The constraint on the total amount of additive can be represented as an integral

$$A \int_{-l}^{0} c(x)dx = M \tag{10}$$

In general, the concentration-viscosity relation will be of the form $\mu = \mu_1 f(c/\hat{c})$, where $\hat{c}$ is a constant with units of concentration and $f(0) = 1$. We assume f is smooth and the inverse functionality is represented as $c = \hat{c}f^{-1}(\mu/\mu_1)$. Substituting into Eq. 10 and nondimensionalizing results in the condition

$$\int_{-L}^{0} f^{-1}(\mu(x))dx = N \tag{11}$$

Here $N = \sqrt{3}k_m M/A\hat{c}$ is the dimensionless amount of material added. This is equivalent to the definition given in Eq. 1. Using the property that the growth constant σ is the minimum possible when $\mu(x)$ is optimal allows the development of a condition relating $\mu(x)$ and $\psi(x)$, see Gorrell and Homsy (1983):

$$(\psi')^2 + 2\beta\psi\psi' + k^2\psi^2 = a\frac{d}{d\mu}\{f^{-1}[\mu(x)]\}. \tag{12}$$

Here a is an undertermined constant.

When solved, Eqs. 6–9 and 11 and 12 will give the viscosity profile which minimizes the growth constant σ for a given wave number k. The largest σ over the entire range of k is the one which is most dangerous. The most dangerous wave number $k_{\max}$ is found by requiring that σ is a maximum, or in terms of β,

$$\frac{\partial\sigma}{\partial k} = \frac{\partial}{\partial k}\left(\frac{k^2}{\beta}\right)_{k=k_{\max}} = 0 \tag{13}$$

Our previous work was for the specific case,

$$\mu = 1 + c \tag{14a}$$

or

$$f^{-1} = \mu - 1 \tag{14b}$$

As can be seen, this choice simplifies the righthand side of Eq. 12. We are concerned in this paper with more general $\mu(c)$.

LARGE-*N* SOLUTION

The unknowns in the problem are the functions $\mu(x)$ and $\psi(x)$ and the constants k, β, a, and L. In addition, the viscosity function $f^{-1}(\mu)$ must be specified. Our previous work indicated, not surprisingly, that the degree of relative stabilization increases with N, and thus any practical process would be carried out under conditions appropriate to large N. When N is large, however, an asymptotic solution can be developed.

Scaling

We have computed numerical solutions to the full problem for different functions $f^{-1}(\mu)$ which indicate that the unknowns always scale with N in the same way. Therefore, when N is large it is assumed

$$L \sim 0(N)$$

$$k \sim 0(N^{-1/3})$$

$$\beta \sim 0(N^{1/3})$$

$$a \sim 0(N^{-2/3})$$

It is appropriate to define 0(1) quantities

$$\tilde{x} = xN^{-1}$$

$$\tilde{L} = LN^{-1}$$

$$\tilde{k} = kN^{1/3}$$

$$\tilde{\beta} = \beta N^{-1/3}$$

$$\tilde{a} = aN^{2/3}$$

The scaled equations for the problem become

$$\psi(\tilde{x})[\tilde{\beta}\bar{\mu}'(\hat{x}) - \tilde{k}^2\mu(x)] + N^{-4/3}[\mu(\tilde{x})\psi(\tilde{x})]' = 0 \tag{15}$$

$$\tilde{\beta}[\psi^2(\tilde{x})]' + \tilde{k}^2\psi^2(\tilde{x}) - \tilde{a}\frac{df^{-1}}{d\mu}[\mu(\tilde{x})] + N^{-4/3}[\psi'(\tilde{x})]^2 = 0 \tag{16}$$

$$N^{-4/3}\mu(0)\psi'(0) = \{\tilde{\beta}[\alpha - \mu(0)] + N^{-2/3}[-\tilde{k}\alpha - (\alpha - 1)\tilde{k}^2\tilde{\beta})]\}\psi(0) \tag{17}$$

$$N^{-2/3}\psi'(-\tilde{L}) = \tilde{k}\psi(-\tilde{L}) \tag{18}$$

$$\mu(-\tilde{L}) = 1 \tag{19}$$

$$\int_{-\tilde{L}}^{0} f^{-1}[\mu(\tilde{x})]d\tilde{x} = 1 \tag{20}$$

$$\frac{\partial}{\partial\tilde{k}}\left(\frac{\tilde{k}^2}{\tilde{\beta}}\right)_{\tilde{k}_{\max}} = 0 \tag{21}$$

The problem is homogeneous in $\psi(\tilde{x})$. Therefore, without any loss of generality we take $\psi(0) = 1$ as a normalization condition. The solution may now be expanded in an asymptotic series in powers of $N^{-2/3}$.

$$\psi(\tilde{x}) = \tilde{\psi}_0(\tilde{x}) + 0(N^{-2/3})$$

$$\mu(\tilde{x}) = \tilde{\mu}_0(\tilde{x}) + N^{-2/3}\tilde{\mu}_1(\tilde{x}) + 0(N^{-4/3})$$

$$\tilde{\beta} = \tilde{\beta}_0 + N^{-2/3}\tilde{\beta}_1 + 0(N^{-4/3})$$

$$\tilde{a} = \tilde{a}_0 + 0(N^{-2/3})$$

$$\tilde{L} = \tilde{L}_0 + N^{-2/3}\tilde{L}_1 + 0(N^{-4/3})$$

$$\tilde{k}_{\max} = \tilde{k}_0 + 0(N^{-2/3})$$

The most dangerous wave number $\tilde{k}_{\max}$, the injection length $\tilde{L}$, the optimal viscosity profile $\mu(\tilde{x})$, and the resulting (minimum) growth constant σ, represent physical quantities which can be measured and varied. We will be concerned primarily with these quantities. Even though higher-order solutions for $\psi_1(\tilde{x})$ and $\tilde{a}$, etc., can be found, they will not be presented because they do not affect these quantities at the lowest orders. In fact, as shown in the Appendix, even the first-order solutions $\tilde{\psi}_o(\tilde{x})$ and $\tilde{a}_o$ come into play only at $0(N^{-4/3})$. Closed-form solutions at this order are impossible because of the nonlinear form of Eq. 16. Only the first term in the $\tilde{k}_{\max}$ expansion can be determined, but that is sufficient to obtain two terms in the growth constant expansion. Substituting the large-N expansions into Eqs. 15–20 results in a sequence of problems. To second order, analytical solutions can be found in terms of integrals involving $f^{-1}(\mu)$. The details are given in the Appendix.

First-Order Solution

The solution for $\tilde{\mu}_o(\tilde{x})$ is

$$\tilde{\mu}_o(\tilde{x}) = \alpha\exp\left(\frac{\tilde{k}^2}{\tilde{\beta}_o}\tilde{x}\right) \tag{22}$$

with

$$\tilde{L}_o = \frac{\tilde{\beta}_o}{\tilde{k}^2}\ln\alpha \tag{23}$$

and

$$\int_{-\tilde{L}_o}^{o} f^{-1}\left[\alpha\exp\left(\frac{\tilde{k}^2}{\tilde{\beta}_o}\tilde{x}\right)\right]d\tilde{x} = 1 \tag{24}$$

Once f^{-1} is specified, Eqs. 23 and 24 can be solved to give $\tilde{k}^2/\tilde{\beta}_o$ and $\tilde{L}_o$, which will be a function only of the mobility ratio, α. For future notational convenience we define $\sigma^* = \tilde{k}^2/\tilde{\beta}_o$.

Second-Order Solution

At $0(N^{-2/3})$ the solution is

$$\mu_1(\tilde{x}) = \left[d - \frac{\tilde{\beta}_1\alpha\sigma^*}{\tilde{\beta}_o}(\tilde{x} + \tilde{L}_o)\right]\exp\{\sigma^*\tilde{x}\} \tag{25}$$

$$\tilde{L}_1 = \frac{d}{\alpha\sigma^*} \tag{26}$$

with the conditions

$$\tilde{\mu}_1(o) = -(\alpha - 1)\tilde{k}^2 - \alpha\sigma^*\tilde{k}^{-1} \tag{27}$$

$$\int_{-\tilde{L}_o}^{o}\tilde{\mu}_1(\tilde{x})\frac{df^{-1}}{d\mu}(\tilde{\mu}_o(\tilde{x}))d\tilde{x} = 0 \tag{28}$$

The correction to the growth constant, $\tilde{\beta}$, and the integration constant d can be found as a function of $\tilde{k}$ by equating Eq. 27 to Eq. 25 evaluated at $x = 0$ and by substituting known results into Eq. 28. At this order, the results are a function of wave number. The most dangerous wave number, $\tilde{k}_o$, can be represented as the root of the equation

$$\left(\frac{\partial d}{\partial\tilde{k}}\right)_{\tilde{k}_o} + 2(\alpha - 1)\tilde{k}_o - \frac{\alpha\sigma^*}{\tilde{k}_o^2} = 0 \tag{29}$$

Evaluating d and $\tilde{\beta}_1$ at $\tilde{k}_o$ in Eqs. 25 and 26 will give the contribution at $0(N^{-2/3})$ to the optimal viscosity profile and injection length which is associated with the fastest growing instability. The growth constant $\sigma_{\max}$ is

$$\sigma_{\max} = \left(\frac{\alpha + 1}{\alpha - 1}\right)\frac{\sigma^*}{N}\left[1 - N^{-2/3}\left(\frac{\tilde{\beta}_1}{\tilde{\beta}_o}\right)_{\tilde{k}_o}\right] + 0(N^{-7/3}) \tag{30}$$

An Example

At this point it is appropriate to illustrate the procedure with a concrete example. As mentioned in the introduction, an exponential viscosity-concentration relation could be a reasonable approximation for the data on polyacrylamide solutions. Therefore, as an example we take $f^{-1}(\mu) = \ln\mu$ and present the results for the optimum viscosity profile.

At first order the solution is given by Eqs. 22–24. Therefore,

$$\tilde{\mu}_0(\tilde{x}) = \alpha\exp\{\sigma^*\tilde{x}\}$$

where

$$\sigma^*\tilde{L}_0 = \ln\alpha \tag{31}$$

$$\int_{-\tilde{L}_o}^{0}(\ln\alpha + \sigma^*\tilde{x})d\tilde{x} = 1 \tag{32}$$

Equations 31 and 32 imply that

$$\sigma^* = \frac{(\ln\alpha)^2}{2}$$

$$\tilde{L}_o = \frac{2}{\ln\alpha}$$

In determining the solution at second order, d and $\tilde{\beta}_1$ must be evaluated. Combining Eqs. 25 and 27,

$$\tilde{\mu}_1(o) = d - \frac{\tilde{\beta}_1}{\tilde{\beta}_o}\alpha\sigma^*\tilde{L}_o \tag{33}$$

In this case $(df^{-1}/d\mu)(\tilde{\mu}_o(\tilde{x})) = \alpha^{-1} \exp\{-\sigma^* \tilde{x}\}$, so Eq. 28 reduces to

$$\int_{-\tilde{L}_o}^{o} \left[d - \frac{\tilde{\beta}_1 \alpha \sigma^*}{\tilde{\beta}_o} (\tilde{x} + \tilde{L}_o)\right] d\tilde{x} = 0 \tag{34}$$

Solving Eqs. 33 and 34 for d and $\tilde{\beta}_1$ in terms of $\tilde{\mu}_1(o)$ yields

$$d = -\tilde{\mu}_1(o)$$

$$\tilde{\beta}_1 = \frac{-2\tilde{\beta}_o \tilde{\mu}_1(o)}{\alpha \sigma^* \tilde{L}_o}$$

From Eqs. 26,

$$\tilde{L}_1 = -\frac{\tilde{\mu}_1(o)}{\alpha \sigma^*}$$

Using these results, the expression detemining $\tilde{k}_o$, Eq. 29 reduces to

$$4(\alpha - 1)\tilde{k}_o - 2\alpha\sigma^* \tilde{k}_o^{-2} = 0$$

from which we find

$$\tilde{k}_o = \left(\frac{\alpha \sigma^*}{2(\alpha - 1)}\right)^{1/3}$$

Now at $\tilde{k} = \tilde{k}_o$,

$$\tilde{\mu}_1(o) = -3 \left(\frac{\alpha \sigma^*}{2}\right)^{2/3} (\alpha - 1)^{1/3} \tag{35}$$

In the expression for d, β_1 and $\tilde{L}_1$ Eq. 35 for $\tilde{\mu}_1(0)$ should be used.

To summarize the results

$$0(1): \quad \tilde{\mu}_o(\tilde{x}) = \alpha \exp\{\sigma^* \tilde{x}\}$$

$$\sigma^* = \frac{(\ln \alpha)^2}{2}$$

$$\tilde{L}_0 = \frac{2}{\ln \alpha}$$

$$0(N^{-2/3}): \quad \tilde{\mu}_1(\tilde{x}) = -3 \left(\frac{\alpha \sigma^*}{2}\right)^{2/3} (\alpha - 1)^{1/3} \left(1 + \frac{2\tilde{x}}{\tilde{L}_o}\right) \exp\{\sigma^* \tilde{x}\}$$

$$\tilde{L}_1 = -3 \left[\frac{(\alpha - 1)}{4\alpha\sigma^*}\right]^{1/3}$$

$$\tilde{k}_0 = \left[\frac{\alpha \sigma^*}{2(\alpha - 1)}\right]^{1/3}$$

Finally, the growth constant associated with this viscosity profile is

$$\sigma_{\max} = \left(\frac{\alpha + 1}{\alpha - 1}\right) \frac{(\ln \alpha)^2}{2N} \left(1 - N^{-2/3} \frac{6}{L_o} \left(\frac{\alpha - 1}{4\alpha\sigma^*}\right)^{1/3}\right] + 0(N^{-7/3}) \tag{36}$$

DISCUSSION

When a large amount of additive is used to stabilize a displacement process, the optimal injection policy can easily be found. To a first approximation the viscosity profile is always an exponential which matches the viscosity of the adjoining fluids. The jump in viscosity at the interface is small, being $0(N^{-2/3})$. This appears to agree with some of the previously mentioned experimental and numerical work, in particular that of Uzoigwe et al., who reported a near viscosity match at the front should be attained. An apparent discrepancy arises, however, in that some experiments suggest, but of course do not prove, that an exponential concentration profile is the best. Our theory, which minimizes the instability due to an *unfavorable mobility* ratio, predicts that an exponential viscosity profile is optimal. An exponential concentration profile will lead to an exponential viscosity profile only if the viscosity is a linear function of concentration or the concentrations of additives used are small. In general, the optimal concentration profile will not be an exponential. For the example considered in the preceding section, our theory predicts that for the viscosity-concentration relation chosen, a linear concentration profile is optimal. Viscosity differences drive the instability in displacement processes and thus there does not appear to be any justifiable reason for an exponential concentration profile to always be optimal. This fact has not been previously emphasized.

The maximum dimensionless growth constant is always positive, but it is an $0(N^{-1})$ quantity. This indicates that $\sigma_{\max}$ can be made arbitrarily small by increasing the amount of additive. It is instructive to consider dimensional values of the maximum growth constant. Retaining only the first term in Eq. 36 gives the dimensionless growth constant as

$$\sigma = \left(\frac{\alpha + 1}{\alpha - 1}\right) \frac{\sigma^* A \hat{c}}{\sqrt{3} k_m M} \tag{37}$$

where k_m is defined in Eq. 4. If we represent the dimensional growth constant as σ_d, then

$$\sigma_d = \frac{3\sigma_m}{2k_m} \left(\frac{\alpha + 1}{\alpha - 1}\right) \frac{\alpha^* A \hat{c}}{m} \cdot \tag{38}$$

Applying Eqs. 4 and 5 for k_m and σ_m results in a simple expression for σ_d,

$$\sigma_d = \frac{A \hat{c} \sigma^* U}{M} \tag{39}$$

The residence time for the interface during the injection process will be $\tau = D/U$, where D is the total distance traveled. Therefore, the total growth of an instability according to linear theory will be

$$\exp\{\sigma_d \tau\} = \exp\left\{\frac{A \hat{c} \sigma^* D}{M}\right\} \tag{40}$$

If the medium for the displacement is fixed and the additive system is specified, then A and $\hat{c}$ are determined. σ^* will depend on the additive and the mobility ratio for the displacement and can be obtained by solving the 0(1) problem (Eqs. 22–24). Note that Eq. 40 predicts that the first approximation to the total growth of an instability for an optimal viscosity profile will be independent of the displacement velocity, U. Furthermore, it can be arbitrarily small by increasing M.

This analysis indicates another aspect of front stabilization with graded mobility. In this analysis it was implicitly assumed that the medium was infinite in cross section. Thus lateral boundaries and the associated wavelength cutoff were ignored. Since this theory predict that k decreases with N such as $N^{-1/3}$, it is possible that the predicted most dangerous wave number will not be physically realizable in bounded systems because of size constraints. This is an additional stabilization mechanism in graded mobility processes which may have been previously overlooked.

From estimates of physical quantities given in the Introduction, it was shown that N is typically large ($\geqslant$2,500). These values can be used to illustrate quantitative features of the solution. For example, if a water flood is performed with no additive to stabilize the front, then the maximum growth constant $\sigma_m = 2.4 \times 10^{-3}\ \text{s}^{-1}$ (from Eq. 3) and the most dangerous wave number $k_m = 11\ \text{cm}^{-1}$ (from Eq. 4). The predicted wavelength is perhaps close to the limit of our continuum theory, but in this example we are interested in the relative changes in k_m due to addition of polymer. The value of σ_m depends upon the displacement velocity, U, being larger for larger U. If an additive for which the viscosity varies exponentially with concentration is used, $[f^{-1}(\mu) = \ln \mu]$ and $N = 2{,}500$, then the corresponding values of the dimensional growth constant σ_d and most dangerous wave number k_d are $\sigma_d = 1.8 \times 10^{-6}\ \text{s}^{-1}$ and $k_d = 1.2\ \text{cm}^{-1}$.

The theory predicts that by injecting solute in an optimal manner, the growth constant of the instability can be decreased by more than a factor of 1,000, while the wave number is decreased by a factor of 10. The critical lateral dimension which would give wavelength cutoff and thus added stability is $(2\pi/1.2)$ cm ~ 5 cm. For the value of N considered here, it is highly unlikely any degree of added stabilization due to finite lateral boundaries would occur in an oil field scale experiment. However, in laboratory scale tests such as core floods and Hele-Shaw experiments, 5 cm is a more representative cross-sectional dimension. In these cases wavelength cutoff stabilization should be considered when analyzing results. This implies that laboratory scale experiments may indicate a greater degree of stabilization than may be obtained in field tests, and therefore may be misleading under some circumstances.

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NOTATION

A = cross-sectional area for flow, m^2
c = concentration of additive, mol/m^3
f = function relating viscosity to concentration, dimensionless
k = disturbance wave number, dimensionless
L = length of graded zone, dimensionless
M = moles of additive
N = parameter defined by Eq. 1, dimensionless
t = time, dimensionless
T = effective interfacial tension, N/m
U = displacement velocity, m/s
x = streamwise distance
y = transverse distance

Greek Letters

α = viscosity ratio, dimensionless
σ = growth constant, dimensionless
ψ = disturbance stream function, dimensionless
μ = fluid viscosity divided by permeability, N·s/m^4

APPENDIX: DETAILS OF SOLUTION

0(1): At lowest order the problem is

$$\tilde{\mu}_o' - \sigma^*\tilde{\mu}_o = 0$$

$$(\tilde{\psi}_o^2)' + \sigma^*\tilde{\psi}_o^2 = \left(\frac{a_o}{\beta_o}\right)\frac{df^{-1}}{d\mu}(\mu_o)$$

$$\tilde{\mu}_o(0) = \alpha \qquad \tilde{\mu}_o(-\tilde{L}_o) = 1$$

$$\tilde{\psi}_o(0) = 1 \qquad \tilde{\psi}_o(-\tilde{L}_o) = 0$$

$$\int_{-\tilde{L}_o}^{o} f^{-1}(\mu_o(\tilde{x})d\tilde{x} = 1$$

The solution (Eqs. 22–24) follows immediately. For completeness, we give the solutions for the disturbance stream function $\psi_o(\tilde{x})$:

$$\tilde{\psi}_o^2(\tilde{x}) = \frac{\tilde{a}_o}{\tilde{\beta}_o} e^{-\sigma^*\tilde{x}} \int_{-\tilde{L}_o}^{\tilde{x}} e^{\sigma^* x}\frac{df^{-1}}{d\mu}[\tilde{\mu}_o(x)]dx$$

$$\frac{\tilde{a}_o}{\tilde{\beta}_o} = \left\{\int_{-L_o}^{o} e^{\sigma^*\tilde{x}}\frac{df^{-1}}{d\mu}[\tilde{\mu}_o(\tilde{x})]d\tilde{x}\right\}^{-1}$$

$0(N^{-2/3})$: At this order the problem is

$$\tilde{\mu}_1' + \left(\frac{\tilde{\beta}_1}{\tilde{\beta}_o}\right)\tilde{\mu}_o' - \sigma^*\mu_1 = 0 \tag{A1}$$

$$\tilde{\beta}_o\tilde{\mu}_1(o) + (\alpha - 1)\tilde{k}^2\tilde{\beta}_o + \alpha\tilde{k} = 0 \tag{A2}$$

$$\int_{-\tilde{L}_o}^{o} \tilde{\mu}_1(\tilde{x})\frac{df^{-1}}{d\mu}\tilde{\mu}_o(\tilde{x})d\tilde{x} = 0$$

$$\tilde{L}_1\tilde{\mu}_o(-\tilde{L}_o) = \tilde{\mu}_1(-\tilde{L}_o) \tag{A3}$$

The solution to Eq. A1 is Eq. 25 where d is a constant of integration. Equation 26 arises directly from the boundary condition in Eq. A3, and Eq. 27 comes from rearranging boundary condition in Eq. A2.

The condition for determining the wave number corresponding to the largest growth constant is

$$\frac{\partial}{\partial\tilde{k}}\left(\frac{\tilde{k}^2}{\tilde{\beta}_o}\right)_{k_{\max}} = 0 \tag{A4}$$

Expanding $\tilde{\beta}$ in its asymptotic series

$$\frac{\tilde{k}^2}{\tilde{\beta}} = \frac{\tilde{k}^2}{\tilde{\beta}_o} - N^{-2/3}\frac{\tilde{k}^2\tilde{\beta}_1}{\tilde{\beta}_o^2} + 0(N^{-4/3})$$

$$= \sigma^*\left(1 - N^{-2/3}\frac{\tilde{\beta}_1}{\tilde{\beta}_o}\right) + 0(N^{-4/3})$$

Since σ^* is not a function $\tilde{k}$, Eq. A4 gives no information at 0(1). At $0(N^{-2/3})$, Eq. A4 implies

$$\frac{\partial}{\partial\tilde{k}}\left(\frac{\tilde{\beta}_1}{\tilde{\beta}_o}\right)_{\tilde{k}_o} = 0 \tag{A5}$$

Evaluating Eq. 25 at $\tilde{x} = 0$ gives

$$\frac{\tilde{\beta}_1}{\tilde{\beta}_o} = \frac{1}{\alpha\sigma^*\tilde{L}_o}[d - \mu_1(o)]$$

From Eq. A5 it follows that

$$\left(\frac{\partial(d - \tilde{\mu}_1(o))}{\partial\tilde{k}}\right)_{\tilde{k}_o} = 0$$

Substituting Eq. 27 for $\tilde{\mu}_1(o)$ results in Eq. 29.

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